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SOME CONSEQUENCES OF THE MORASS AND DIAMOND *

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Morasses were invented by Ronald B. Jensen and were shown by him to exist in the universe of constructible sets. Using Morasses and sometimes other principles such as Diamond (◇), which also hold in the constructible universe, he has solved many cases of the gap n, 2 cardinal conjectures of model theory and several other problems as well.

Here we give the definition and some of the basic properties of the gap 2 Morass, and then show, assuming Zermelo–Fraenkel set theory with the Axiom of Choice, that the existence of a gap 2 Morass and the principle Diamond (◇) imply

(1) $\omega_{\alpha+2} \not\to [\omega_{\alpha+1} + \omega_\alpha]^2_{\aleph_{\alpha+1}}$,

(2) a combinatorial principle which extends a principle of Prikry,

(3) a result concerning free sets of ordinals related to problems posed by Hajnal and Máté.

0. Introduction

0.1. Morasses were invented and proved to exist in the universe of constructible sets by R.B. Jensen in order to show, among other things, that the gap n, 2 cardinal conjectures of model theory are true in the universe of constructible sets. An early proof of the gap 2, 2 cardinal conjecture where the smaller cardinal is regular is given in K.J. Devlin's notes [2]. Other cases occur in Jensen's notes [9]. Using Morasses and the principle ◇ (Diamond), which also holds in the constructible universe, Jensen has also proved that there is an $\aleph_2$ universal linear order and that a combinatorial principle of K. Prikry holds. R. Laver has shown that the Morass implies the Mess of Jech. Here we show in ZFC (Zermelo–Fraenkel set theory with the Axiom of Choice) that a gap 2

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Morass and $\Diamond$ imply

(1) $\omega_{\alpha+2} \nrightarrow [\omega_{\alpha+1} + \omega_\alpha]^2_{\aleph_{\alpha+1}}$,[1]

(2) an extension of the above combinatorial principle of Prikry,

(3) a result concerning free sets of ordinals related to some problems posed in [7].

The proofs that Morasses exist in the constructible universe are lengthy and complicated. Although a proof that the gap 2 Morasses exist may be found in [2], for a rigorous development of the preliminaries, see Jensen's paper [8]. Proofs of the existence of various gap n Morasses occur in Jensen's notes [9].

Our exposition however requires no knowledge of constructibility, and, we hope, is entirely self-contained. We are concerned only with consequences of gap 2 Morasses (the gap n Morasses are more complicated) and the well known principle $\Diamond$. In Section 1 we give a definition of gap 2 Morasses and state or derive several of their elementary or frequently used properties. In Section 2 we define the arrow relations we are concerned with, discuss what is known and not known about arrow relations similar to the ones we are concerned with, state $\Diamond$, and then give, using $\Diamond(\omega_1)$ and an ω_1 Morass, one of the easiest consequences of Morasses, viz., $\omega_2 \nrightarrow (\omega_1 : \omega)^2_2$. In Section 3 we prove the general case of our extension of Prikry's combinatorial principle, and discuss its genesis. In Section 4 we derive $\omega_{\alpha+2} \nrightarrow [\omega_{\alpha+1} + \omega_\alpha]^2_{\aleph_{\alpha+1}}$ from this extended principle. Also in Section 4, we give the relevant definitions, state some problems of Hajnal and Máté and deduce our result related to these problems.

0.2. *Notations and conventions*

We assume Zermelo–Fraenkel set theory with the Axiom of Choice, and adopt the usual conventions and definitions of set theory. To be specific, we adopt the notation, conventions and definitions of [11, pp. 5–11] unless the contrary is indicated below.

Throughout, we assume κ is a cardinal $\geqslant \omega$, κ^+ the least cardinal greater than κ, π, usually with subscripts, is a mapping, and other lower case greek letters, except ω, φ and ψ, range over ordinals. An ordinal is the set of preceding ordinals, i.e., $\alpha = \{\beta : \beta < \alpha\} = \{\beta : \beta \in \alpha\}$. The usual

[1] This problem was suggested to me by Richard Laver.

convention is to write ω_α when we think of it as an ordinal and $\aleph_\alpha$ when we think of it as a cardinal, but as far as we are concerned, for all α, $\omega_\alpha = \aleph_\alpha$. Lower case latin letters will usually range over ordinals, except f and g, which range over functions. 0 = the empty set = the least ordinal.

For any x, y, we write $\langle x, y\rangle$ for the ordered pair whose first element is x and whose second element is y. For any X,

$$\mathrm{Dom}(X) = \{x\colon (\exists y(\langle x, y\rangle \in X)\},$$

$$\mathrm{Rng}(X) = \{y\colon (\exists x(\langle x, y\rangle \in X)\},$$

$$X^{-1} = \{\langle y, x\rangle\colon \langle x, y\rangle \in X\},$$

the inverse of X. For any X, Y,

$$X\,''\,Y = \{z\colon (\exists x \in X)(\exists y \in Y)(x = \langle y, z\rangle)\},$$

$$X \circ Y = \{\langle y, x\rangle\colon (\exists z)(\langle y', z\rangle \in Y \text{ and } \langle z, x\rangle \in X)\}.$$

If f and g are functions, $f \circ g$ is that function h defined by $h(y) = f(g(y))$ for $y \in \mathrm{Dom}(g)$ and $g(y) \in \mathrm{Dom}(f)$. We frequently write, for example, $\langle a_b : b \in X\rangle$ to mean a function f with $\mathrm{Dom}(f) = X$ and for $b \in X$, $f(b) = a_b$. Similarly for such expressions as $\langle \pi_{\nu\tau}\colon \nu \leftarrow \tau, \nu,\tau \in S\rangle$. We write $f : X \to Y$ to mean that f is a function whose domain is X and whose range is included in Y. $f \upharpoonright X$, the restriction of f to X, is $\{\langle x, y\rangle \in f\colon x \in X\}$. An enumeration of a set X is a 1-1 function whose domain is an ordinal and whose range is X. So an enumeration is what others have called an enumeration without repetitions. If X is a set of ordinals, then type(X), the type of X, is that unique ordinal α such that there is a 1-1 order preserving map of α onto X.

If X, Y are sets of ordinals and α is an ordinal, we will write $X < Y$ to mean for each $\xi \in X$ and each $\eta \in Y$, $\xi < \eta$; $X < \alpha$ to mean for each $\xi \in X$, $\xi < \alpha$; etc.

A tree T is a partial order such that for each $x \in \mathrm{Dom}(T) \cup \mathrm{Rng}(T)$,

$$\{\langle y, z\rangle \in T\colon \langle y, x\rangle \in T \text{ and } \langle z, x\rangle \in T\}$$

is a linear order, i.e., any two predecessors of x are comparable. A tree can have more than one minimal element.

Hereafter when we speak of Morasses we mean gap 2 Morasses.

In the definition of Morasses we will mention certain models $\mathfrak{M}_\tau$ and mappings $\pi_{\nu\tau}$ of $\mathfrak{M}_\nu$ into $\mathfrak{M}_\tau$ and say that $\pi_{\nu\tau}$ is a Σ_0 elementary embedding of $\mathfrak{M}_\nu$ into $\mathfrak{M}_\tau$. To describe these models and mappings we need briefly a few of the basic notions of model theory. So we assume the usual definitions of model, language, formula, satisfaction, isomorphism, etc., that we require as presented in [1]. The models $\mathfrak{M}_\tau$ are of the form $\langle M_\tau, =, \epsilon, R_0, R_1, ...\rangle$, where $\tau \subseteq M_\tau$. Given such a model $\mathfrak{M}_\tau$, let φ be a formula of the language appropriate to $\mathfrak{M}_\tau$. We say φ is a Σ_0 formula if all the quantifiers occurring in φ are bounded, that is, all are of the form $(\exists x)(x \in y \;\&\; ...)$ or of the form $(\forall x)(x \in y \to ...)$. We can define Σ_0 formulas by induction on the length of formulas as follows: Atomic formulas are Σ_0 formulas; if φ and ψ are Σ_0 formulas then $\neg\varphi, \varphi \;\&\; \psi$, $(\exists x)(x \in y \;\&\; \varphi)$, and $(\forall x)(x \in y \to \varphi)$ are Σ_0 formulas. We may abbreviate these last two formulas by $(\exists x \in y)\varphi$ and $(\forall x \in y)\varphi$, respectively. A relation is said to be a Σ_0 relation on the model $\mathfrak{M}_\tau$, or Σ_0 $(\mathfrak{M}_\tau)$, if it is definable on $\mathfrak{M}_\tau$ by a Σ_0 formula. That is, a relation R is Σ_0 $(\mathfrak{M}_\tau)$ if and only if there are a Σ_0 formula φ and parameters $a_1, ..., a_n \in \mathfrak{M}_\tau$ such that for all $x_1, ..., x_m$, we have

$$R(x_1, ..., x_m) \quad \text{iff} \quad \mathfrak{M}_\tau \models \varphi[x_1, ..., x_m, a_1, ..., a_n] \ .$$

We say that $\mathfrak{M}_\tau$ is a Σ_0 elementary submodel of $\mathfrak{M}'_\tau$ if $\mathfrak{M}_\tau$ is a submodel of $\mathfrak{M}'_\tau$ and for every Σ_0 formula φ and all $x_1, ..., x_m \in \mathfrak{M}_\tau$, we have $\mathfrak{M}_\tau \models \varphi[x_1, ..., x_m]$ if and only if $\mathfrak{M}'_\tau \models \varphi[x_1, ..., x_m]$. We say that $\pi_{\nu\tau}$ is a Σ_0 elementary embedding of $\mathfrak{M}_\nu$ into $\mathfrak{M}_\tau$ if $\pi_{\nu\tau}$ is an isomorphism of $\mathfrak{M}_\nu$ onto a Σ_0 elementary submodel of $\mathfrak{M}_\tau$. For the basic properties and discussion of the general case of Σ_n elementary submodels etc., see [8].

1. Morasses. Definition and basic properties

A Morass is a tree, whose nodes are ordinals, together with a set of mappings which satisfy certain commutativity and continuity conditions as well as certain relations to other ordinals not nodes of the tree. The following definition is almost verbatim from [2].

Definition. A *κ-Morass* (κ regular, $\kappa > \omega$) is defined as follows.

Let $\mathscr{A}$ be a set of pairs $\langle \alpha, \nu \rangle$ of ordinals with $0 < \alpha \leqslant \kappa$ and $\nu < \kappa^+$

such that $\alpha < \nu$ and if $\langle\alpha,\nu\rangle, \langle\alpha',\nu'\rangle \in \mathscr{A}$ and $\alpha < \alpha'$, then $\nu < \alpha'$. Let

$$A = \{\alpha \in \kappa \cup \{\kappa\}: (\exists\nu)(\langle\alpha,\nu\rangle \in \mathscr{A})\} ,$$

$$S = \{\nu \in \kappa^+: (\exists\alpha)(\langle\alpha,\nu\rangle \in \mathscr{A})\} ,$$

$$S_\alpha = \{\nu \in \kappa^+ : \langle\alpha,\nu\rangle \in \mathscr{A}\} \quad \text{for } \alpha \in A .$$

Then for $\alpha, \alpha' \in A, \alpha < \alpha'$ we have $\alpha < S_\alpha < \alpha' < S_{\alpha'}$ and for $\nu \in S$ there is a unique $\alpha \in A$ such that $\langle\alpha,\nu\rangle \in \mathscr{A}$. Let α_ν = the unique $\alpha \in A$ such that $\langle\alpha,\nu\rangle \in \mathscr{A}$, for $\nu \in S$. For each $\alpha \in A$ we require that S_α be closed in $\sup(S_\alpha)$. Let $\leftarrow$ be a tree on S such that if $\nu \leftarrow \tau$, then $\alpha_\nu < \alpha_\tau$. For each $\tau \in S$, let $\mathfrak{M}_\tau = \langle M_\tau, \in, ...\rangle$ be a transitive structure with $\mathrm{Ord} \cap M_\tau = \tau$ such that for $\nu \in S_{\alpha_\tau} \cap \tau$ we have $\mathfrak{M}_\nu \subseteq \mathfrak{M}_\tau$. For each $\nu, \tau \in S$ with $\nu \leftarrow \tau$, let $\pi_{\nu\tau}$ be a mapping from $M_\nu \cup \{\nu\}$ into $M_\tau \cup \{\tau\}$ such that $\pi_{\nu\tau}(\nu) = \tau$, $\pi_{\nu\tau} \upharpoonright \alpha_\nu = \mathrm{id} \upharpoonright \alpha_\nu$, $\pi_{\nu\tau}(\alpha_\nu) = \alpha_\tau$, and $\pi_{\nu\tau}$ is a Σ_0 elementary embedding of $\mathfrak{M}_\nu$ into $\mathfrak{M}_\tau$, i.e., we assume that $\mathfrak{M}_\nu, \mathfrak{M}_\tau$ are models of the same type and if $\varphi(v_1, ..., v_n)$ is any formula of the language appropriate to $\mathfrak{M}_\nu$ all of whose quantifiers are bounded and if $a_1, ..., a_n \in M_\nu$, then

$$\mathfrak{M}_\nu \models \varphi[a_1, ..., a_n] \quad \text{iff} \quad \mathfrak{M}_\tau \models \varphi[\pi_{\nu\tau}(a_1), ..., \pi_{\nu\tau}(a_n)] .$$

Also $\pi_{\nu\tau}$ maps $S_{\alpha_\nu} \cap \nu$ into $S_{\alpha_\tau} \cap \tau$ in a way which preserves order and the properties of being a limit point and of being an immediate successor. I.e., if λ is a limit point in $S_{\alpha_\nu} \cap \nu$, then $\pi_{\nu\tau}(\lambda)$ is a limit point in $S_{\alpha_\tau} \cap \tau$; if $\xi, \eta \in S_{\alpha_\nu} \cap \nu$ and ξ immediately precedes η in $S_{\alpha_\nu} \cap \nu$, then $\pi_{\nu\tau}(\xi)$ immediately precedes $\pi_{\nu\tau}(\eta)$ in $S_{\alpha_\tau} \cap \tau$; and if ξ is the least member of $S_{\alpha_\nu} \cap \nu$, then $\pi_{\nu\tau}(\xi)$ is the least member of $S_{\alpha_\tau} \cap \tau$. The system of maps is commutative in the sense that if $\xi \leftarrow \nu \leftarrow \tau$, then $\pi_{\xi\tau}(x) = \pi_{\nu\tau}(\pi_{\xi\nu}(x))$ for all $x \in M_\xi$. Then the structure

$$\mathfrak{M} = \langle \mathscr{A}, \leftarrow, \langle \pi_{\nu\tau}: \nu \leftarrow \tau\rangle, \langle \mathfrak{M}_\tau : \tau \in S\rangle\rangle$$

is a κ-Morass iff in addition the following seven conditions are satisfied.

(i) $\kappa = \max(A) = \sup(A - \{\kappa\})$ and $\kappa^+ = \sup(S_\kappa)$.

(ii) If $\bar\tau \leftarrow \tau$ and $\bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau$ and $\nu = \pi_{\bar\tau\tau}(\bar\nu)$, then $\bar\nu \leftarrow \nu$ and $\pi_{\bar\nu\nu} \upharpoonright M_{\bar\nu} = \pi_{\bar\tau\tau} \upharpoonright M_{\bar\nu}$.

(iii) $\{\alpha_\nu : \nu \leftarrow \tau\}$ is closed in α_τ.

(iv) If τ is not maximal in S_{α_τ}, then $\{\alpha_\nu : \nu \leftarrow \tau\}$ is unbounded in α_τ.

(v) If $\{\alpha_\nu : \nu \leftarrow \tau\}$ is unbounded in α_τ, then $\mathfrak{M}_\tau = \bigcup_{\nu\leftarrow\tau} \pi''_{\nu\tau}\mathfrak{M}_\nu$.

(vi) If $\bar\tau$ is a limit point of $S_{\alpha_{\bar\tau}}$ and $\bar\tau \leftarrow \tau$ and $\lambda = \sup_{\bar\nu<\bar\tau} \pi_{\bar\tau\tau}(\bar\nu)$, then $\bar\tau \leftarrow \lambda$ and $\pi_{\bar\tau\lambda} \upharpoonright M_{\bar\tau} = \pi_{\bar\tau\tau} \upharpoonright M_{\bar\tau}$.

(vii) If $\bar{\tau}$ is a limit point of $S_{\alpha_{\bar{\tau}}}$ and $\bar{\tau} \leftarrow \tau$ and

$$\tau = \sup_{\bar{\nu} < \bar{\tau}} \pi_{\bar{\tau}\tau}(\bar{\nu}) , \qquad \alpha \in \bigcap_{\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}} \{\alpha_\eta : \bar{\nu} \preceq \eta \preceq \pi_{\bar{\tau}\tau}(\bar{\nu})\} ,$$

then there is $\tau' \in S_\alpha$ such that $\bar{\tau} \preceq \tau' \preceq \tau$.

This completes the definition.

Following is the comment on this definition in [2].

"For each $\alpha \in A$, S_α is an 'approximation' to S_κ. Since $\alpha < \kappa$ implies $|S_\alpha| < \kappa$ and $|S_\kappa| = \kappa^+$, this 'approximation' cannot be thought of as anything remotely connected with unions of chains. However, the tree $\leftarrow$ on S is the device which makes this 'approximation' idea precise. Note that by (iv), if α is a successor in A, $|S_\alpha| = 1$, so that we only obtain 'good approximations' at limit stages in A."

We next list some immediate consequences of the definition and a few propositions required later.

(a) Suppose τ is an immediate successor of $\bar{\tau}$ in $\leftarrow$. Then $\alpha_{\bar{\tau}} < \bar{\tau} < \alpha_\tau < \tau$, $\{\alpha_\nu : \nu \leftarrow \tau\} = \{\alpha_\nu : \nu \preceq \bar{\tau}\}$ and so $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_{\bar{\tau}} < \bar{\tau} < \alpha_\tau$, i.e., $\sup\{\alpha_\nu : \nu \leftarrow \tau\}$ is bounded in α_τ. Then by (iv) τ is maximal in S_{α_τ}.

(b) If τ is minimal in $\leftarrow$, then $\{\alpha_\nu : \nu \leftarrow \tau\} = 0$ and $\{\alpha_\nu : \nu \leftarrow \tau\}$ is again bounded in α_τ, so by (iv) again, τ is maximal in S_{α_τ}.

(c) Suppose $\{\alpha_\nu : \nu \leftarrow \tau\}$ is bounded in α_τ. If $\{\alpha_\nu : \nu \leftarrow \tau\} = 0$, then τ is minimal in $\leftarrow$. If $\{\alpha_\nu : \nu \leftarrow \tau\} \neq 0$, then $0 < \sup\{\alpha_\nu : \nu \leftarrow \tau\} < \alpha_\tau$ and so by (iii) there is $\bar{\tau} \leftarrow \tau$ such that $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_{\bar{\tau}}$. τ must be an immediate successor of $\bar{\tau}$ in $\leftarrow$ since if not, we would have $\bar{\tau} \leftarrow \tau' \leftarrow \tau$ for some τ' and $\alpha_{\bar{\tau}} < \bar{\tau} < \alpha_{\tau'} < \tau'$ which would contradict the fact that $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_{\bar{\tau}}$.

Putting (a), (b) and (c) together we have

(d) τ is minimal in $\leftarrow$ iff $\{\alpha_\nu : \nu \leftarrow \tau\} = 0$.

τ is an immediate successor in $\leftarrow$ iff $\{\alpha_\nu : \nu \leftarrow \tau\} \neq 0$ and $\{\alpha_\nu : \nu \leftarrow \tau\}$ is bounded in α_τ.

τ is an immediate successor of $\bar{\tau}$ in $\leftarrow$ iff $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_{\bar{\tau}}$ and $\bar{\tau} \leftarrow \tau$.

τ is a limit point in $\leftarrow$ iff τ is not minimal in $\leftarrow$ and τ is not an immediate successor in $\leftarrow$ iff $\{\alpha_\nu : \nu \leftarrow \tau\}$ is unbounded in α_τ iff $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_\tau$.

Thus there are three mutually exclusive possibilities:

(1) $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = 0$ and τ is minimal in $\leftarrow$;

(2) $0 < \sup\{\alpha_\nu : \nu \leftarrow \tau\} < \alpha_\tau$ and τ has an immediate predecessor $\bar{\tau}$ in $\leftarrow$ and $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_{\bar{\tau}}$;

(3) $\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_\tau$ and τ is a limit point in $\leftarrow$.

(e) Let m be the map defined by $m(\nu) = \alpha_\nu$ for $\nu \in S$. Let b be a branch in the tree $\leftarrow$. Then $m \restriction b$ is 1–1, order preserving, and continuous. (It is continuous in the sense that for τ a limit point in $\leftarrow$, $m(\tau) = \sup\{m(\nu) : \nu \leftarrow \tau\}$.)

(f) τ not maximal in S_{α_τ} implies

$$\{\alpha_\nu : \nu \leftarrow \tau\} \text{ is unbounded in } \alpha_\tau,$$

which is equivalent to

$$\sup\{\alpha_\nu : \nu \leftarrow \tau\} = \alpha_\tau,$$

which is equivalent to

$$\tau \text{ is a limit point in } \leftarrow,$$

which implies

$$\mathfrak{M}_\tau = \bigcup_{\nu \leftarrow \tau} \pi''_{\nu\tau} \mathfrak{M}_\nu .$$

(g) By (iv), if α is a successor in A, then $\mathrm{Card}(S_\alpha) = 1$, i.e., there is exactly one $\tau \in S_\alpha$ which is therefore both maximal and minimal in S_α and which is either minimal in $\leftarrow$ or an immediate successor in $\leftarrow$.

(h) Let τ be a limit point in $\leftarrow$. Then $\{\alpha_\nu : \nu \leftarrow \tau\}$ is unbounded in α_τ and $\mathfrak{M}_\tau = \bigcup_{\nu \leftarrow \tau} \pi''_{\nu\tau} \mathfrak{M}_\nu$. Let $\nu \in S_{\alpha_\tau} \cap \tau$. Now $S_{\alpha_\tau} \cap \tau \subseteq M_\tau$, so for some $\bar{\tau} \leftarrow \tau$ and some $\bar{\nu} \in \bar{\tau}$ we have $\nu = \pi_{\bar{\tau}\tau}(\bar{\nu})$.[2] From the fact that $S_{\alpha_{\bar{\tau}}}$ is closed in $\sup(S_{\alpha_{\bar{\tau}}})$ and since $\pi_{\bar{\tau}\tau}$ preserves order and the property of being an immediate successor in $S_{\alpha_{\bar{\tau}}}$, it follows that $\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$. Thus

Proposition 1. *If τ is a limit point in $\leftarrow$ and $\nu \in S_{\alpha_\tau} \cap \tau$, then there exist $\bar{\tau}, \bar{\nu}$ such that $\bar{\tau} \leftarrow \tau$, $\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$ and $\nu = \pi_{\bar{\tau}\tau}(\bar{\nu})$.*

(j) The following proposition is mainly what condition (vii) in the Morass definition is all about. In the sequel, condition (vii) is only used in the form of this proposition.

[2] The relation of being an ordinal is a Σ_0 relation.

Proposition 2. *Let τ immediately succeed $\bar{\tau}$ in $\leftarrow$, let τ be a limit point in S_{α_τ} and let*

$$\tau = \sup\{\pi_{\bar{\tau}\tau}(\bar{\nu}) : \bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}\}.$$

For $\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$, let $\eta(\bar{\nu})$ = the least η such that $\bar{\nu} \leftarrow \eta \leftarrow \pi_{\bar{\tau}\tau}(\bar{\nu})$. Then

$$\sup\{\alpha_{\eta(\bar{\nu})} : \bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}\} = \alpha_\tau .$$

Proof. Since τ is a limit point in S_{α_τ}, it follows that $\bar{\tau}$ is a limit point in $S_{\alpha_{\bar{\tau}}}$. And for $\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$, $\pi_{\bar{\tau}\tau}(\bar{\nu})$ is a limit point in $\leftarrow$. So for such $\bar{\nu}$, clearly $\eta(\bar{\nu})$ exists and we have

$$\bar{\nu} \leftarrow \eta(\bar{\nu}) \leftarrow \pi_{\bar{\tau}\tau}(\bar{\nu}), \qquad \alpha_{\bar{\nu}} < \bar{\nu} < \alpha_{\eta(\bar{\nu})} < \eta(\bar{\nu}) < \alpha_\tau .$$

So $\sup\{\alpha_{\eta(\bar{\nu})} : \bar{\nu} \in S_{\alpha_\tau} \cap \bar{\tau}\} \leq \alpha_\tau$. Suppose $\sup\{\alpha_{\eta(\bar{\nu})} : \bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}\} = \alpha$ and $\alpha < \alpha_\tau$. Let $\bar{\xi}, \bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$ and let $\bar{\xi} < \bar{\nu}$. Let $\xi = \pi_{\bar{\tau}\tau}(\bar{\xi})$, $\nu = \pi_{\bar{\tau}\tau}(\bar{\nu})$. Then $\bar{\xi} \leftarrow \eta(\bar{\xi}) \leftarrow \xi$ and $\bar{\nu} \leftarrow \eta(\bar{\nu}) \leftarrow \nu$.[3]

Now $\pi_{\bar{\nu}\eta(\bar{\nu})}(\bar{\xi})$ is on the branch from $\bar{\xi}$ to ξ. [This follows from the facts that $\pi_{\bar{\tau}\tau} \restriction M_{\bar{\nu}} = \pi_{\bar{\nu}\nu} \restriction M_{\bar{\nu}}$, $\bar{\xi} \in M_{\bar{\nu}}$, and $\pi_{\bar{\nu}\nu}(x) = \pi_{\eta(\bar{\nu})\nu}(\pi_{\bar{\nu}\eta(\bar{\nu})}(x))$ for all $x \in M_{\bar{\nu}}$.] $\eta(\bar{\xi})$ is by definition on the branch from $\bar{\xi}$ to ξ, and since $\leftarrow$ is a tree, $\eta(\bar{\xi})$ and $\pi_{\bar{\nu}\eta(\bar{\nu})}(\bar{\xi})$ are comparable. Then by the definition of $\eta(\bar{\xi})$ as the least thing on the branch from $\bar{\xi}$ to ξ we have $\eta(\bar{\xi}) \underline{\leftarrow} \pi_{\bar{\nu}\eta(\bar{\nu})}(\bar{\xi})$. Since $\pi_{\bar{\nu}\eta(\bar{\nu})}(\bar{\xi})$ is not maximal in $S_{\alpha_{\eta(\bar{\nu})}}$, it follows that it is a limit point in the tree and thus $\eta(\bar{\xi}) \leftarrow \pi_{\bar{\nu}\eta(\bar{\nu})}(\bar{\xi})$. So $\alpha_{\eta(\bar{\xi})} < \alpha_{\eta(\bar{\nu})}$.

$$\alpha = \sup\{\alpha_{\eta(\bar{\nu})} : \bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}\} = \sup\{\alpha_{\eta(\bar{\nu})} : \bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau} \;\&\; \bar{\xi} < \bar{\nu}\}$$

$$= \sup\{\alpha_{\xi'} : \xi' = \pi_{\bar{\nu}\eta(\bar{\nu})}(\bar{\xi}) \;\&\; \bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau} \;\&\; \bar{\xi} < \bar{\nu}\} .$$

Since $\{\alpha_{\xi'} : \xi' \leftarrow \xi\}$ is closed in $\alpha_\xi = \alpha_\tau$ (by property (iii) in the Morass definition), it follows that there is ξ' such that $\bar{\xi} \leftarrow \xi' \leftarrow \xi$ with $\alpha_{\xi'} = \alpha$. Therefore, by property (vii) in the Morass definition, there is $\tau' \in S_\alpha$ such that $\bar{\tau} \underline{\leftarrow} \tau' \underline{\leftarrow} \tau$. Since τ immediately succeeds $\bar{\tau}$ in $\leftarrow$, it follows that either $\tau' = \bar{\tau}$ or $\tau' = \tau$. But both of these are impossible since $\alpha_{\bar{\tau}} < \alpha = \alpha_{\tau'} < \alpha_\tau$. Contradiction! Thus $\alpha = \alpha_\tau$.

[3] See Fig. 1. In this and subsequent figures we make the following conventions. Members of A, say α, are on the bottom horizontal line. For $\alpha \in A$ we represent S_α as points on a vertical line up from α. We so to speak rotate S_α 90° counterclockwise. A solid line with an arrowhead on its left end between two points indicates that the left hand point is the immediate predecessor of the right hand point in the tree.

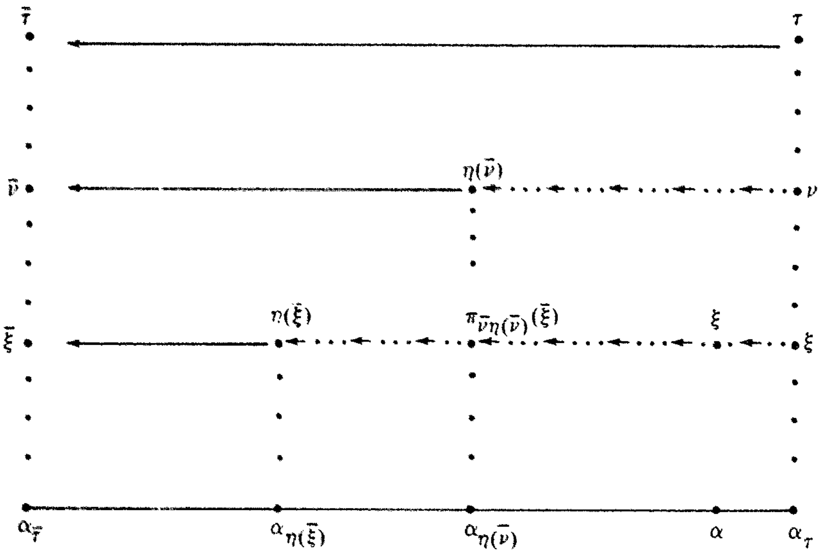

Fig. 1.

(k) the next proposition, easy as it is, is given here for the record since it is used explicitly at least once later, and it may be used implicitly elsewhere.

Proposition 3. *Let* $\tau_1, \tau_2 \in S, \tau_1 \neq \tau_2$. *Then*

(a) *if* τ_1 *is a limit point in* $\leftarrow$, *then* $\{\nu: \nu \leftarrow \tau\} \neq \{\nu: \nu \leftarrow \tau_2\}$;

(b) *If* $\alpha_{\tau_1} = \alpha_{\tau_2}$, *then* $\{\nu: \nu \leftarrow \tau_1\} \neq \{\nu: \nu \leftarrow \tau_2\}$.

Proof. (a) Let $\tau_1, \tau_2 \in S, \tau_1 \neq \tau_2$. Suppose τ_1 is a limit point in $\leftarrow$ and $\{\nu: \nu \leftarrow \tau_1\} = \{\nu: \nu \leftarrow \tau_2\}$. Then τ_2 is a limit point in $\leftarrow$ and $\alpha_{\tau_1} = \alpha_{\tau_2}$ since each equals $\sup\{\alpha_\nu: \nu \leftarrow \tau_1\}$. We may assume $\tau_1 < \tau_2$. Then since $\tau_1 \in S_{\alpha_{\tau_2}} \cap \tau_2$ is a limit point in $\leftarrow$, by Proposition 1 there exist $\nu \leftarrow \tau_2$ and $\nu' \in S_{\alpha_\nu} \cap \nu$ such that $\pi_{\nu\tau_2}(\nu') = \tau_1$. Then $\nu' \leftarrow \tau_1$. But $\nu \leftarrow \tau_1$ also and ν, ν' are incomparable. Contradiction, since $\leftarrow$ is a tree.

(b) follows from (a) since at most the maximal point (if any) of any S_α can be a successor point in $\leftarrow$. Thus two different points $\tau_1, \tau_2 \in S$ can have the same set of $\leftarrow$ predecessors only if either (i) they are both minimal in $\leftarrow$ or (ii) $\alpha_{\tau_1} \neq \alpha_{\tau_2}$ and they are both immediate successors (and then necessarily of the same point).

2. Morass and diamond imply $\omega_2 \nrightarrow (\omega_1 : \omega)^2_2$

2.1. *Some partition relations*

The relation $\omega_2 \nrightarrow (\omega_1 : \omega)^2_2$ is a member of a class of combinatorial relations invented and developed extensively by Erdös, Hajnal, and Rado [6, 5, 3]. They are called partition relations or arrow relations. We will give the definitions and a few basic properties of some of them, but for a general presentation, see [6].

If X is any set, $\langle I_\xi : \xi < \kappa \rangle$ is called a partition of X if $\bigcup_{\xi<\kappa} I_\xi = X$ and for all $\xi < \eta < \kappa$ we have $I_\xi \cap I_\eta = 0$. Here κ is any cardinal. If X is any set and κ any cardinal, $[X]^\kappa$ denotes the set of κ-element subsets of X. We will be concerned with partitions of $[X]^2$ which establish the negation of corresponding positive arrow relations. In general, an expression with a "$\nrightarrow$" is the negation of the corresponding expression with a "$\rightarrow$".

The meaning of $\omega_2 \rightarrow (\omega_1 : \omega)^2_2$ is: For every R, G such that $R \cap G = 0$, $R \cup G = \{\{\alpha, \beta\} : \alpha < \beta < \omega_2\}$ there exist A, B such that $A \subseteq \omega_2, B \subseteq \omega_2$, type$(A) = \omega_1$, type$(B) = \omega$, $A < B$ and either (i) $\{\{\alpha,\beta\} : \alpha \in A \ \&\ \beta \in B\} \subseteq R$ or (ii) $\{\{\alpha,\beta\} : \alpha \in A \ \&\ \beta \in B\} \subseteq G$.

Sometimes a partition is spoken of as a coloring. In this case we say that members of R are red and members of G are green.

Then $\omega_2 \nrightarrow (\omega_1 : \omega)^2_2$ means: There is a partition $\langle R, G \rangle$ of $[\omega_2]^2$ such that for every A, B with $A \subseteq \omega_2$, $B \subseteq \omega_2$, type$(A) = \omega_1$, type$(B) = \omega$, and $A < B$ there exist $\alpha, \alpha' \in A$, $\beta, \beta' \in B$ such that $\{\alpha, \beta\} \in R$ and $\{\alpha', \beta'\} \in G$.

The more general relation, $\alpha \rightarrow (\beta : \eta)^2_\delta$ where α, β, η are ordinals and δ is a cardinal, means: For every partition $\langle I_\xi : \xi < \delta \rangle$ of $[\alpha]^2$ there exist $\xi < \delta, A, B$ such that $A \subseteq \alpha, B \subseteq \alpha$, type$(A) = \beta$, type$(B) = \eta$, $A < B$ and $\{\{a,b\} : a \in A \ \&\ b \in B\} \subseteq I_\xi$.

Two more widely known partition relations studied by the above mathematicians and others, and related to the above are $\alpha \rightarrow (\beta)^\gamma_\delta$ and $\alpha \rightarrow [\beta]^\gamma_\delta$, where α, β are ordinals, γ, δ are cardinals.

$\alpha \rightarrow (\beta)^\gamma_\delta$ means: For every partition $\langle I_\xi : \xi < \delta \rangle$ of $[\alpha]^\gamma$ there exist $\xi < \delta, B \subseteq \alpha$, type$(B) = \beta$ such that $[B]^\gamma \subseteq I_\xi$.

$\alpha \rightarrow [\beta]^\gamma_\delta$ means: For every partition $\langle I_\xi : \xi < \delta \rangle$ of $[\alpha]^\gamma$ there exist $\xi < \delta, B \subseteq \alpha$, type$(B) = \beta$ such that $[B]^\gamma \cap I_\xi = 0$.

In terms of colors, $\alpha \rightarrow (\beta)^\gamma_\delta$ [respectively, $\alpha \rightarrow [\beta]^\gamma_\delta$] means: However you may color the γ element subsets of α with δ colors there will always be a subset B of α of type β and some color ξ such that every [respectively, no] γ element subset of B has color ξ.

The definition of $\alpha \to [\beta : \eta]^2_\delta$ is analogous: For every partition $\langle I_\xi : \xi < \delta \rangle$ of $[\alpha]^2$ there exist $\xi < \delta$, A, B such that $A \subseteq \alpha$, $B \subseteq \alpha$, $\text{type}(A) = \beta$, $\text{type}(B) = \eta$, $A < B$ and $\{\{a, b\} : a \in A \ \& \ b \in B\} \cap I_\xi = 0$.

It is immediate from the definitions that

$\alpha \to (\beta)^\gamma_2$ and $\alpha \to [\beta]^\gamma_2$ are equivalent,
$\alpha \to (\beta : \eta)^2_2$ and $\alpha \to [\beta : \eta]^2_2$ are equivalent,
$\alpha \to (\beta)^\gamma_\delta$ implies $\alpha \to [\beta]^\gamma_\delta$,
$\alpha \to (\beta : \eta)^2_\delta$ implies $\alpha \to [\beta : \eta]^2_\delta$,
$\alpha \to (\beta + \eta)^2_\delta$ implies $\alpha \to (\beta : \eta)^2_\delta$,
$\alpha \to [\beta + \eta]^2_\delta$ implies $\alpha \to [\beta : \eta]^2_\delta$.

For any of the above relations, a valid relation remains valid if α is increased, β or η decreased. A valid parenthesis relation remains valid if δ is decreased; a valid square bracket relation remains valid if δ is increased. (We mean of course only relations with "$\to$"; for relations with "$\not\to$", a valid relation remains valid if α decreases, β increases etc.)

In this section we will show that if there is an ω_1-Morass and if $\diamondsuit(\omega_1)$ holds, then $\omega_2 \not\to [\omega_1 : \omega]^2_2$. In a later section we will show that if there is an $\omega_{\alpha+1}$-Morass and $\diamondsuit(\omega_{\alpha+1})$ holds, then $\omega_{\alpha+2} \not\to [\omega_{\alpha+1} : \omega_\alpha]^2_{\aleph_{\alpha+1}}$. Then, since in the constructible universe there is an $\omega_{\alpha+1}$-Morass and $\diamondsuit(\omega_{\alpha+1})$ holds, we will have shown that in the constructible universe

$$\omega_{\alpha+2} \not\to [\omega_{\alpha+1} + \omega_\alpha]^2_{\aleph_{\alpha+1}}.$$

2.2. *Related results*

It had been known that $\omega_2 \not\to [\omega_1 + \omega]^2_{\aleph_1}$ is relatively consistent, for Erdös and Hajnal state in [4]: "With a slight generalization [of a result of Prikry], Hajnal proved

$$\text{Con(ZF)} \quad \text{implies} \quad \text{Con}(\text{ZFC} + \text{GCH} + \omega_2 \not\to [\omega_1 + \omega]^2_{\aleph_1})$$

and also

$$\text{Con(ZF)} \quad \text{implies} \quad \text{Con}(\text{ZFC} + \text{GCH} + \omega_2 \not\to [\omega_1 + 2]^2_{\aleph_0})\text{''}.$$

They continue: "Now the following would be interesting to see: is $\omega_2 \to (\omega_1 + \omega)^2_2$ or even $\omega_2 \to (\xi)^2_2$ for $\xi < \omega_2$ consistent with GCH?...

We proved [GCH implies] $\omega_2 \to (\omega_1 + n)^2_2$ for $n < \omega$ several years ago, but we do not know at present: does GCH imply $\omega_2 \to (\omega_1 + 2)^2_3$?"

A simple form of Ramsey's theorem (which is the seed from which these partition relations grew) is: $\omega \to (\omega)^2_2$. It is known that $\omega_1 \to (\omega + n)^2_2, n \in \omega$ [6]. It is easy to show that $\omega_2 \to (\omega_1 : 2)^2_n, n \in \omega$.

Erdös and Hajnal [4] also ask: "Assume GCH. Is it true that for every $\alpha < \omega_2$ there is $\xi < \omega_3$ such that $\xi \to (\alpha)^2_2$? GCH certainly implies e.g. $\omega_2 \cdot 2 \to (\omega_1 + \omega)^2_2$. The following result of Shelah is a generalization of GCH implies $\omega_2 \to (\omega_1 + n)^2_2$.

Theorem (Shelah). Assume GCH. Let $\aleph_\alpha, \aleph_\beta$ be regular, $\beta + 2 \leqslant \alpha$. Then $\omega_{\alpha+1} \to (\omega_\alpha + \omega_\beta)^2_2$."

2.3. *The principle* $\diamondsuit(\kappa)$

Let κ be a regular cardinal $> \omega$. Then the principle $\diamondsuit(\kappa)$ is known to hold in the universe of constructible sets; moreover it holds in $L[A]$, the universe of sets constructible from A, where A is any subset of κ. The proof of this is attributed to "Everyone" in [10].

$\diamondsuit(\kappa)$: There is a sequence $\langle Y_\xi : \xi < \kappa \rangle$ such that $Y_\xi \subseteq \xi$ for $\xi < \kappa$ and for every $X \subseteq \kappa$, the set $\{\xi : X \cap \xi = Y_\xi\}$ is stationary in κ; i.e., for every closed unbounded $C \subseteq \kappa$, $C \cap \{\xi : X \cap \xi = Y_\xi\} \neq 0$.

There are many equivalent formulations of $\diamondsuit(\kappa)$ and many consequences and relations with other principles. See for example [10]. For one thing $\diamondsuit(\kappa^+)$ implies $2^\kappa = \kappa^+$. Probably the most famous consequence of $\diamondsuit(\kappa^+)$, due to Jensen, is the negation of the Souslin hypothesis, i.e., there exists a normal Souslin tree on κ^+.

In the proof of $\omega_2 \nrightarrow [\omega_1 : \omega]^2_2$ from $\diamondsuit(\omega_1)$ and an ω_1 Morass, we use Diamond to get so-called distinguished subsets B_α of S_α for $\alpha \in A$, $\alpha < \omega_1$ (S_α, A are as in the Morass definition), at most one per S_α, such that for any $B \subseteq S_{\omega_1}$ with type$(B) = \omega$ and $\tau = \sup(B)$, there is an unbounded set of α's in A such that $\pi''_{\bar{\tau}\tau} B_\alpha = B$, where $\bar{\tau} = \sup(B_\alpha)$ and $\bar{\tau} \leftarrow \tau$. Thus for $\alpha \in A, \alpha < \omega_1$, there is at most a countable number of B_α's and their projections by means of the maps $\pi_{\nu\tau}$ for $\nu \leftarrow \tau, \alpha_\tau = \alpha$, to be dealt with.

2.4. *Proof that an ω_1-Morass and $\diamondsuit(\omega_1)$ imply $\omega_2 \nrightarrow [\omega_1 : \omega]_2^2$*

Let $\mathcal{M} = \langle \mathcal{A}, \leftarrow, \langle \pi_{\nu\tau} : \nu \leftarrow \tau \rangle, \langle \mathfrak{M}_\tau : \tau \in S \rangle \rangle$ be an ω_1-Morass. Let S, A, S_α for $\alpha \in A$ and α_τ for $\tau \in S$ be as in the definition of Morasses. It is enough to color the two-element subsets of S_{ω_1} since type$(S_{\omega_1}) = \omega_2$.

We first use $\diamondsuit(\omega_1)$ to define distinguished subsets B_α for $\alpha \in A - \{\omega_1\}$. Let $B \subseteq S_{\omega_1}$, type$(B) = \omega$, $\tau = \sup(B)$. Since ω_1 is regular and Card$(\{\nu : \nu \leftarrow \tau\}) = \omega_1$, it follows from Proposition 1 that there is a least $\nu_0 \leftarrow \tau$ such that $B \subseteq \pi''_{\nu_0\tau} S_{\alpha_{\nu_0}}$. For ν such that $\nu_0 \mathrel{\underline{\leftarrow}} \nu \leftarrow \tau$ let $X_\nu = (\pi_{\nu\tau}^{-1})''B$. Then type $(X_\nu) = \omega$, $\nu = \sup(X_\nu)$ and for $\nu_0 \mathrel{\underline{\leftarrow}} \bar\nu \leftarrow \nu \leftarrow \tau$ we have $\pi''_{\bar\nu\nu} X_{\bar\nu} = X_\nu$. Let $X = \bigcup_{\nu_0 \mathrel{\underline{\leftarrow}} \nu \leftarrow \tau} X_\nu$. Let $\langle Y_\xi : \xi < \omega_1 \rangle$ be the sequence given by $\diamondsuit(\omega_1)$. Then by $\diamondsuit(\omega_1)$, $\{\xi : X \cap \xi = Y_\xi\}$ meets every closed unbounded subset of ω_1. For each $\bar\nu \leftarrow \tau$, $C_{\bar\nu} = \{\alpha_\nu : \bar\nu \leftarrow \nu \leftarrow \tau\}$ is a closed unbounded subset of ω_1. So for each such $\bar\nu$ there is $\alpha \in C_{\bar\nu}$ such that $Y_\alpha = X \cap \alpha$. In other words, given B, τ, X as above there exist arbitrarily large $\alpha \in A - \{\omega_1\}$ and $\nu \leftarrow \tau$ such that $\alpha = \alpha_\nu$ and $Y_\alpha = X \cap \alpha$.

Now define B_α for $\alpha \in A - \{\omega_1\}$ as follows.

Case 1. There exist B, τ, ν_0, X, ν such that $B \subseteq S_{\omega_1}$, type$(B) = \omega$, $\tau = \sup(B)$, $\nu_0 \mathrel{\underline{\leftarrow}} \nu \leftarrow \tau$, ν_0 is the least ordinal $\leftarrow \tau$ such that $\pi''_{\nu_0\tau} S_{\alpha_{\nu_0}} \supseteq B$, $X = \bigcup_{\nu_0 \mathrel{\underline{\leftarrow}} \bar\nu \leftarrow \tau} X_{\bar\nu}$, $\alpha = \alpha_\nu$ and $Y_\alpha = X \cap \alpha$. We define B_α in this case by $B_\alpha = \bigcup_{\nu_0 \mathrel{\underline{\leftarrow}} \bar\nu \leftarrow \nu} \pi''_{\bar\nu\nu} X_{\bar\nu}$.

How do we know that this definition is independent of the choice of B (and thus τ, ν_0, X etc.)? Suppose we had $B_1 \neq B_2$, τ_i, X^i, ν_i, etc. defined as above such that $\alpha = \alpha_{\nu_1} = \alpha_{\nu_2}$ and $Y_\alpha = X^1 \cap \alpha = X^2 \cap \alpha$. $Y_\alpha = \bigcup_{\nu_0 \mathrel{\underline{\leftarrow}} \bar\nu \leftarrow \nu_1} X_{\bar\nu}$, and this is a disjoint union since $X_{\bar\nu} = Y_\alpha \cap S_{\alpha_{\bar\nu}}$. Furthermore, $\bar\nu$ can be recovered from $X_{\bar\nu}$ since $\bar\nu = \sup(X_{\bar\nu})$. That is, $\{\bar\nu : \nu_0 \mathrel{\underline{\leftarrow}} \bar\nu \leftarrow \nu_1\}$ can be recovered from (or is determined by) Y_α, since for $\bar\alpha \in A$, $\bar\alpha < \alpha$, either $Y_\alpha \cap S_{\bar\alpha} = 0$, or $Y_\alpha \cap S_{\bar\alpha}$ has type ω. So

$$\{\bar\nu : \nu_0 \mathrel{\underline{\leftarrow}} \bar\nu \leftarrow \nu_1\} = \{\bar\nu : (\exists \bar\alpha < \alpha)(\bar\alpha \in A \;\&\; S_{\bar\alpha} \cap Y_\alpha \neq 0 \;\&\; \bar\nu = \sup(S_{\bar\alpha} \cap Y_\alpha))\} .$$

Then by Proposition 3, $\nu_1 = \nu_2$ since $\alpha_{\nu_1} = \alpha_{\nu_2}$ and ν_1 and ν_2 have the same predecessors in $\leftarrow$. Thus the definition is unambiguous.

Case 2 (not Case 1). Set $B_\alpha = 0$. [Note that in case 1, $B_\alpha = X_\nu$ when Y_{α_ν} is as above, since in such a case Y_{α_ν} uniquely determines ν and then $X_\nu = \pi''_{\bar\nu\nu} X_{\bar\nu}$ for any $\bar\nu \leftarrow \nu$ with $S_{\alpha_{\bar\nu}} \cap Y_{\alpha_\nu} \neq 0$].

Thus the B_α's have been defined so that for any $B \subseteq S_{\omega_1}$ with type$(B) = \omega$ and $\tau = \sup(B)$ there are arbitrarily large $\alpha \in A - \{\omega_1\}$ and $\nu \leftarrow \tau$ such that $\pi''_{\nu\tau} B_\alpha = B$.

We now use the distinguished sets B_α to define a partition of $\bigcup_{\alpha \in A} [S_\alpha]^2$ into two parts. Let $\alpha \in A - \{\omega_1\}$. Let

$$\mathcal{B}'_\alpha = \{H\colon (\exists \nu \in S_\alpha)(\exists \bar\nu \preceq \nu)(H = \pi''_{\bar\nu\nu} B_{\alpha_{\bar\nu}} \;\&\; H \text{ is infinite})\}.^4$$

Let

$$\mathcal{B}_\alpha = \{H\colon (\exists H' \in \mathcal{B}'_\alpha)(\exists \xi < \sup(H'))\,(H = H' - \xi)\}.$$

$\mathcal{B}_\alpha$ is then the set of co-initial segments of members of $\mathcal{B}'_\alpha$. $\mathcal{B}_\alpha$ is at most countable and each member of $\mathcal{B}_\alpha$ has type ω. Let $\langle H_i^\alpha\colon i \in \omega\rangle$ be an enumeration of $\mathcal{B}_\alpha$. (We may write H_i instead of H_i^α when there should be no confusion.) We can choose $\rho_i, \gamma_i \in H_i$ all distinct as follows. Let $\rho_0, \gamma_0 \in H_0$, $\rho_0 \neq \gamma_0$. Supposing $\rho_0, \ldots, \rho_n, \gamma_0, \ldots, \gamma_n$ have been chosen, all distinct, $\rho_i, \gamma_i \in H_i$ for $0 \leq i \leq n$, we can choose $\rho_{n+1}, \gamma_{n+1} \in H_{n+1}$, $\rho_{n+1} \neq \gamma_{n+1}$ and different from all ρ_i, γ_i for $i \leq n$ since H_{n+1} is countably infinite. Suppose we have done this for each $\alpha \in A - \{\omega_1\}$.

For $\nu \in S$ we will define a partition of $S_{\alpha_\nu} \cap \nu$ into two pieces R_ν, G_ν such that

(1) if $\bar\nu \leftarrow \nu$, then $\pi''_{\bar\nu\nu} R_{\bar\nu} \subseteq R_\nu$ and $\pi''_{\bar\nu\nu} G_{\bar\nu} \subseteq G_\nu$,

(2) if ν is a limit point in $\leftarrow$ then

$$R_\nu = \bigcup_{\bar\nu \leftarrow \nu} \pi''_{\bar\nu\nu} R_{\bar\nu}, \qquad G_\nu = \bigcup_{\bar\nu \leftarrow \nu} \pi''_{\bar\nu\nu} G_{\bar\nu},$$

(3) if ν has $\bar\nu$ as its immediate predecessor in $\leftarrow$, then we have two cases,

(3.1) $\bar\nu = \rho_n$ for some n, where $\rho_n \in H_n^{\alpha_{\bar\nu}}$ as defined above, then

$$R_\nu = (\pi''_{\bar\nu\nu} R_{\bar\nu}) \cup [(S_{\alpha_\nu} \cap \nu) - \pi''_{\bar\nu\nu} S_{\alpha_{\bar\nu}}], \qquad G_\nu = \pi''_{\bar\nu\nu} G_{\bar\nu},$$

(3.2) otherwise,

$$R_\nu = \pi''_{\bar\nu\nu} R_{\bar\nu}, \qquad G_\nu = (\pi''_{\bar\nu\nu} G_{\bar\nu}) \cup [(S_{\alpha_\nu} \cap \nu) - \pi''_{\bar\nu\nu} S_{\alpha_{\bar\nu}}],$$

(4) if ν is minimal in $\leftarrow$, then $R_\nu = S_{\alpha_\nu} \cap \nu$, $G_\nu = 0$.

In case (3), $(S_{\alpha_\nu} \cap \nu) - \pi''_{\bar\nu\nu} S_{\alpha_{\bar\nu}}$ may be viewed as the set of "new" points, since the "old" points, namely $\pi''_{\bar\nu\nu} S_{\alpha_{\bar\nu}}$, have already been taken care of

[4] $\pi_{\nu\nu} = \mathrm{id} \upharpoonright M_\nu$.

at stage $\bar{\nu}$ or earlier. So we will have colored the "new" points red with ν if $\bar{\nu}$ is ρ_n for some n; otherwise the new points will be colored green with ν.

The definition is by induction on $\nu \in S$. Simply define R_ν, G_ν by (4), (3) or (2) according to whether ν is minimal, an immediate successor, or a limit point in $\leftarrow$, respectively. Then it is easy to check that for all $\nu \in S$, $\langle R_\nu, G_\nu \rangle$ is a partition of $S_{\alpha_\nu} \cap \nu$ as desired. (The only case that is not immediate from the definitions is when ν is a limit point in $\leftarrow$. But then we have it right away from the induction hypotheses, from Proposition 1 and the fact that the maps $\pi_{\bar{\nu}\nu}$ are 1-1 and commutative.)

Let

$$R = \{\{\nu, \tau\}: \nu, \tau \in S_{\omega_1} \ \&\ \nu \in R_\tau\},$$

$$G = \{\{\nu, \tau\}: \nu, \tau \in S_{\omega_1} \ \&\ \nu \in G_\tau\}.$$

To complete the proof, we show that for any B, Z, if $B \subseteq S_{\omega_1}$, $Z \subseteq S_{\omega_1}$, type$(B) = \omega$, type$(Z) = \omega_1$, $Z < B$, then there exist $b, b' \in B$, $z \in Z$ with $\{b, z\} \in R$ and $\{b', z\} \in G$. Let B, Z be as above. Let $\tau = \sup(B)$, $\theta = \sup(Z)$. Let $\nu_0 \leftarrow \tau$ such that $\pi''_{\nu_0\tau} B_{\alpha_{\nu_0}} = B$ and such that $\theta \in \pi''_{\nu_0\tau} S_{\alpha_{\nu_0}}$. Let $z \in Z - \pi''_{\nu_0\tau} S_{\alpha_{\nu_0}}$. (Since type$(Z) = \omega_1$ and $S_{\alpha_{\nu_0}}$

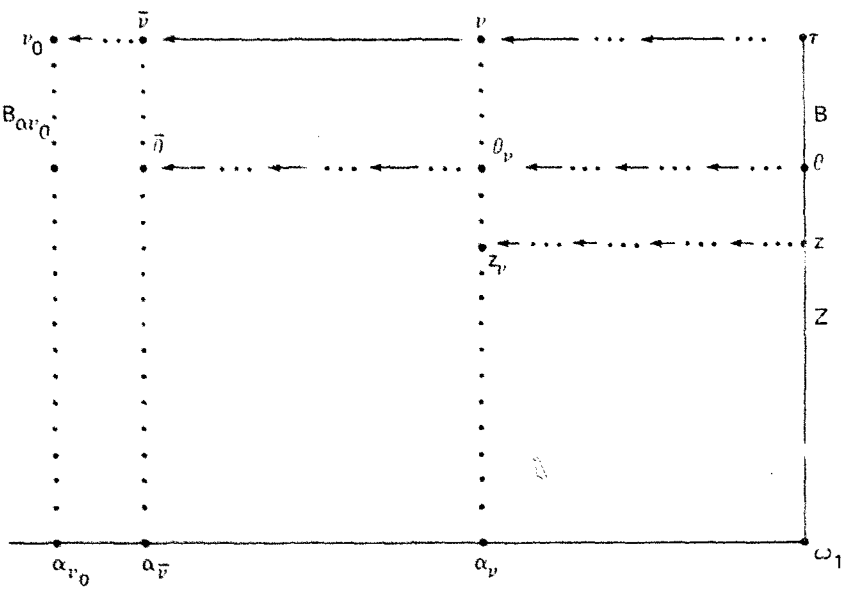

Fig. 2.

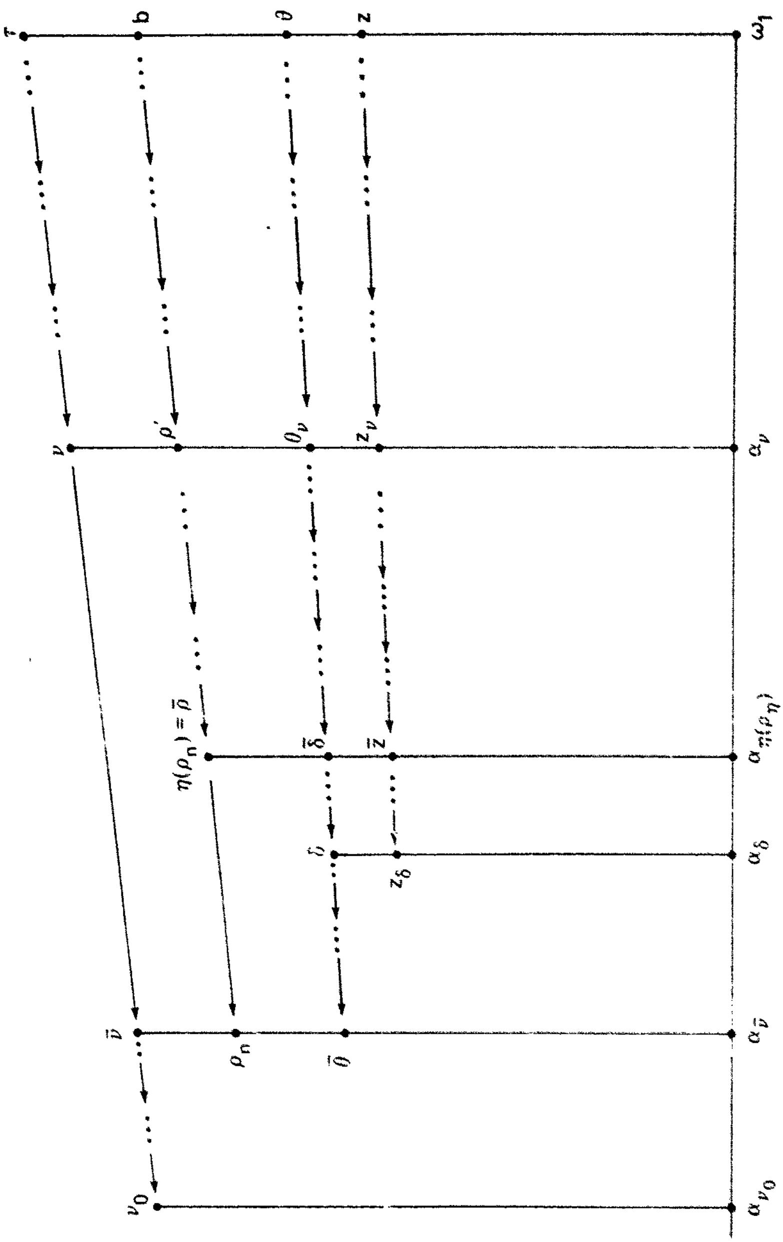

Fig. 3.

is countable, there is such a z.) Then there is a least ν such that $\nu_0 \leftarrow \nu \leftarrow \tau$ and $z \in \pi''_{\nu\tau} S_{\alpha_\nu}$. ν must have an immediate predecessor in $\leftarrow$, say $\bar\nu$. Then $\nu_0 \leftarrow \bar\nu \leftarrow \nu \leftarrow \tau$ and there exist $z_\nu, \theta_\nu \in S_{\alpha_\nu} \cap \nu$, $z_\nu \in \theta_\nu$, $\bar\theta \in S_{\alpha_{\bar\nu}} \cap \bar\nu$ such that $\pi_{\nu\tau}(z_\nu) = z$ for all $\bar z \in S_{\alpha_{\bar\nu}} \cap \bar\nu$, $\pi_{\bar\nu\nu}(\bar z) \neq z_\nu$, $\pi_{\bar\nu\nu}(\bar\theta) = \theta_\nu$, $\pi_{\nu\tau}(\theta_\nu) = \theta$, $\pi_{\bar\nu\tau}(\bar\theta) = \theta$, etc. See Fig. 2.

Now θ_ν is a limit point in $\leftarrow$ and $z_\nu \in S_{\alpha_\nu} \cap \theta_\nu$. So there is a least δ such that $\bar\theta \leftarrow \delta \leftarrow \theta_\nu$ and such that for some $z_\delta \in S_{\alpha_\delta}$, $\pi_{\delta\theta_\nu}(z_\delta) = z_\nu$. Note that $\delta \neq \bar\theta$ and $\delta \neq \theta_\nu$. So $\alpha_{\bar\nu} < \alpha_\delta < \alpha_\nu$. Since $\nu_0 \underline{\leftarrow} \bar\nu \leftarrow \nu \leftarrow \tau$, $\tau = \sup(B)$ and $\pi''_{\nu_0\tau} B_{\alpha_{\nu_0}} = B$, it follows that $\nu_0 = \sup(B_{\alpha_{\nu_0}})$, $\bar\nu = \sup(\pi''_{\nu_0\bar\nu} B_{\alpha_{\nu_0}})$, $\nu = \sup(\pi''_{\bar\nu\nu}\pi''_{\nu_0\bar\nu} B_{\alpha_{\nu_0}})$, etc. So in particular $\nu = \sup\{\pi_{\bar\nu\nu}(\xi) \colon \xi \in S_{\alpha_{\bar\nu}} \cap \bar\nu\}$. For $\xi \in S_{\alpha_{\bar\nu}} \cap \bar\nu$, let $\eta(\xi)$ be the least member of the branch from ξ to $\pi_{\bar\nu\nu}(\xi)$ greater than ξ. By Proposition 2 then, we have $\alpha_\nu = \sup\{\alpha_{\eta(\xi)} \colon \xi \in S_{\alpha_{\bar\nu}} \cap \bar\nu\}$.

Now $\pi''_{\nu_0\bar\nu} B_{\alpha_{\nu_0}} \in \mathcal{B}'_{\alpha_{\bar\nu}}$ and all its co-initial segments are in $\mathcal{B}_{\alpha_{\bar\nu}}$. So there is a co-initial segment of $\pi''_{\nu_0\bar\nu} B_{\alpha_{\nu_0}}$, say $H_n^{\alpha\bar\nu}$, where $\rho_n, \gamma_n \in H_n \subseteq S_{\alpha_{\bar\nu}} \cap \bar\nu$, as defined above, and

$$(*) \quad \alpha_\delta < \alpha_{\eta(\rho_n)} < \alpha_\nu, \qquad (**) \quad \alpha_\delta < \alpha_{\eta(\gamma_n)} < \alpha_\nu .$$

Let $\rho' = \pi_{\bar\nu\nu}(\rho_n)$. Let $\bar\rho = \eta(\rho_n)$ and $\bar\delta = \pi_{\rho_n\bar\rho}(\bar\theta)$. $\bar\delta$ is on the branch from $\bar\theta$ to θ_ν and so is δ. By (*), $\delta \leftarrow \bar\delta$. Let $\bar z = \pi_{\delta\bar\delta}(z_\delta)$. See Fig. 3. Then $\bar z \in (S_{\alpha_{\bar\rho}} \cap \bar\rho) - \pi''_{\rho_n\bar\rho} S_{\alpha_{\bar\nu}}$. So by (3) in the definition of $\langle R_\zeta, G_\zeta \colon \zeta \in S\rangle$ we have $\bar z \in R_{\bar\rho}$. Then $z_\nu \in R_{\rho'}$ and letting $b = \pi_{\nu\tau}(\rho')$ we have $z \in R_b$, i.e., $\{b, z\} \in R$. $b \in B$ since $b = \pi_{\bar\nu\tau}(\rho_n)$, $\rho_n \in H_n$ and $\pi''_{\bar\nu\tau} H_n \subseteq B$. Similarly, using γ_n and (**), we see that there is $b' \in B$ such that $\{b', z\} \in G$. This completes the proof.

3. Morass and Diamond imply an extension of a principle of Prikry

3.1. *Introduction*

In his paper [12], Prikry showed the relative consistency of the partition relation

$$\binom{\aleph_2}{\aleph_1} \nrightarrow \begin{bmatrix}\aleph_0\\ \aleph_1\end{bmatrix}_{\aleph_1} ,$$

defined below, by means of a combinatorial principle which he denoted (*) and which he showed to hold in a certain forcing model. Later Jensen, using an ω_1-Morass and $\Diamond(\omega_1)$ showed that (*) holds in L. In this section we show that a Morass and Diamond imply an extension (*) of this principle.

We first define the partition relations and the combinatorial principle Prikry considered.

$$\binom{\aleph_2}{\aleph_1} \nrightarrow \begin{pmatrix}\aleph_0 & \aleph_0\\ \aleph_1 & \aleph_1\end{pmatrix}$$

means: there is a partition of $\omega_2 \times \omega_1$ into two classes I_0, I_1 such that there are no A, B, with $A \subseteq \omega_2$, $B \subseteq \omega_1$, $\mathrm{Card}(A) = \aleph_0$, $\mathrm{Card}(B) = \aleph_1$ and $A \times B \subseteq I_0$ or $A \times B \subseteq I_1$.

$$\binom{\aleph_2}{\aleph_1} \nrightarrow \begin{bmatrix}\aleph_0\\ \aleph_1\end{bmatrix}_{\aleph_1}$$

means: there is a partition of $\omega_2 \times \omega_1$ into $\aleph_1$ classes I_γ, $\gamma \in \omega_1$ such that for every A, B, $A \subseteq \omega_2$, $B \subseteq \omega_1$, $\mathrm{Card}(A) = \aleph_0$, $\mathrm{Card}(B) = \aleph_1$ and for every $\gamma \in \omega_1$, $(A \times B) \cap I_\gamma \neq 0$.

The principle (*) is: There are sets $A_{\beta\gamma} \subseteq \omega_1$, $\beta \in \omega_2$, $\gamma \in \omega_1$ such that for every $\beta, \gamma, \bar\gamma$, $\gamma \neq \bar\gamma$ implies $A_{\beta\gamma} \cap A_{\beta\bar\gamma} = 0$; for every β, $\bigcup_{\gamma\in\omega_1} A_{\beta\gamma} = \omega_1$: and for every $f : \omega_2 \to \omega_1$ with type $(\mathrm{dom}(f)) = \omega$,

$$\mathrm{Card}(\omega_1 - \bigcup_{\beta\in\mathrm{dom}(f)} A_{\beta f(\beta)}) \leqslant \aleph_0 .$$

In this case we say that $\bigcup_{\beta\in\mathrm{dom}(f)} A_{\beta f(\beta)}$ is almost all of ω_1. In general we say that X is almost all of Y if $\mathrm{Card}(Y - X) < \mathrm{Card}(Y)$.

Prikry states two other principles, which he denotes by $\Delta(\aleph_2, \aleph_1)$ and $\Box(\aleph_2, \aleph_1)$. [5] He notes that $\Box(\aleph_2, \aleph_1)$ implies

$$\binom{\aleph_2}{\aleph_1} \nrightarrow \begin{bmatrix}\aleph_0\\ \aleph_1\end{bmatrix}_{\aleph_1},$$

which implies

$$\binom{\aleph_2}{\aleph_1} \nrightarrow \begin{bmatrix}\aleph_0 & \aleph_0\\ \aleph_1 & \aleph_1\end{bmatrix}.$$

and shows that (*), $\Delta(\aleph_2, \aleph_1)$, and $\Box(\aleph_2, \aleph_1)$ are equivalent.

[5] Not to be confused with Jensen's principle $\Box(\kappa_1)$.

We will show that if there is a κ^+ Morass and if $\diamond(\kappa^+)$ holds, then the following principle holds.

$(\hat{*})(\kappa^+)$. There are sets $A_{\beta\gamma} \subseteq \beta$ for $\beta \in \kappa^{++}$ and $\gamma \in \beta$ such that for every $\beta, \gamma, \bar{\gamma}, \gamma \neq \bar{\gamma}$ implies $A_{\beta\gamma} \cap A_{\beta\bar{\gamma}} = 0$; for every β, $\bigcup_{\gamma\in\beta} A_{\beta\gamma} = \beta$; and for every $f: \kappa^{++} \to \kappa^{++}$ with type $(\mathrm{dom}(f)) = \kappa$ and $f(\beta) < \beta$, for every $\theta < \mathrm{dom}(f)$,

$$\mathrm{Card}(\theta - \bigcup_{\beta\in\mathrm{dom}(f)} A_{\beta f(\beta)}) \leqslant \kappa\ .$$

There are analogues $\hat{\Delta}$, $\hat{\Box}$ of Δ and $\Box$ which are equivalent to $(\hat{*})$.

3.2. *Proof that a κ^+-Morass & $\diamond(\kappa^+)$ imply $(\hat{*})(\kappa^+)$*

Let $\mathfrak{M}$ be a κ^+-Morass, A, S, S_α for $\alpha \in A$, $\mathfrak{M}_\tau$, $\pi_{\nu\tau}$ for $\nu, \tau \in S$, $\nu \leftarrow \tau$, be as in the definition of Morasses. It is sufficient to construct $\langle A_{\nu\gamma} \subseteq S_{\kappa^+} \cap \nu: \nu \in S_{\kappa^+}, \gamma \in S_{\kappa^+} \cap \nu\rangle$ since type $(S_{\kappa^+}) = \kappa^{++}$. We first use $\diamond(\kappa^+)$ to get, for $\alpha \in A - \{\kappa^+\}$, at most one $f_\alpha : S_\alpha \to S_\alpha$ with type $(\mathrm{dom}(f_\alpha)) = \kappa$ and $f_\alpha(x) < x$ for all $x \in \mathrm{dom}(f_\alpha)$. The f_α are called distinguished functions. They will be such that for any $f: S_{\kappa^+} \to S_{\kappa^+}$ with type $(\mathrm{dom}(f)) = \kappa$ and $f(x) < x$ for all $x \in \mathrm{dom}(f)$, there exist $\tau > \mathrm{dom}(f)$ and arbitrarily large $\nu \leftarrow \tau$ with $f_{\alpha_\nu} \subseteq S_{\alpha_\nu} \times S_{\alpha_\nu}$ and $\pi''_{\nu\tau} f_{\alpha_\nu} = f$. That is, for each $\bar{\nu} \leftarrow \tau$ there is a larger $\nu \leftarrow \tau$ such that

$$f = \{\langle \pi_{\nu\tau}(x), \pi_{\nu\tau}(f_{\alpha_\nu}(x))\rangle : x \in \mathrm{dom}(f_{\alpha_\nu})\}\ ,$$

and in fact

$$f_{\alpha_\nu}(x) = y \quad \text{iff} \quad f(\pi_{\nu\tau}(x)) = \pi_{\nu\tau}(y) \qquad \text{for all } x, y \in S_{\alpha_\nu}.$$

This will be done by coding up these functions as subsets.

So let $f: S_{\kappa^+} \to S_{\kappa^+}$, type $(\mathrm{dom}(f)) = \kappa$, $f(x) < x$ for $x \in \mathrm{dom}(f)$. There is a Σ_0 relation, GP, Gödel's pairing function, such that

$$\forall x,y,z\ [\mathrm{GP}(x, y, z) \Rightarrow \mathrm{Ord}(x)\ \&\ \mathrm{Ord}(y)\ \&\ \mathrm{Ord}(z)]\ ,$$

(hereafter, x, y and z are assumed to be ordinals)

$$\forall x,y\ \exists! z\ [\mathrm{GP}(x, y, z)\ \&\ x \leqslant z\ \&\ y \leqslant z]\ ,$$

$$\forall z\ \exists! x\ \exists! y\ \mathrm{GP}(x, y, z)\ ,$$

$$\forall x,y,\kappa\,[x,\, y < \kappa \;\&\; \mathrm{GP}(x,\, y,\, z) \Rightarrow z < \kappa]\,.$$

Let $B = \{z\colon \mathrm{GP}(x, y, z) \;\&\; f(x) = y\}$. Let τ be the least ordinal of which B is a subset. Then $f \subseteq \tau \times \tau$. Let ν_0 be the least ν such that $\nu \leftarrow \tau$ & $B \subseteq \pi''_{\nu\tau}\nu$. (Use from Proposition 1 the facts that κ^+ is regular and that type $(\{\nu\colon \nu \leftarrow \tau\}) = \kappa^+$). For $\nu_0 \preceq \nu \leftarrow$, $x \in \mathrm{dom}(f)$, $z \in \tau$,

$$\mathrm{GP}(x, f(x), z) \quad \text{iff} \quad \mathrm{GP}(\pi^{-1}_{\nu\tau}(x),\, \pi^{-1}_{\nu\tau}(f(x)),\, \pi^{-1}_{\nu\tau}(z))$$

and for $\bar{z} \in \nu$,

$$\mathrm{GP}(\pi^{-1}_{\nu\tau}(x),\, \pi^{-1}_{\nu\tau}(f(x)),\, z) \qquad \text{iff} \quad \mathrm{GP}(x, f(x), \pi_{\nu\tau}(\bar{z}))$$

etc., since $\pi_{\nu\tau}$ is a Σ_0 elementary embedding. For $\nu_0 \preceq \nu \leftarrow \tau$ let $X_\nu = \pi^{-1}_{\nu\tau}{}''B$. Let $X = \bigcup_{\nu_0 \preceq \nu \leftarrow \tau} X_\nu$. Let $\langle Y_\xi : \xi \in \kappa^+\rangle$ be the sequence given by $\diamondsuit(\kappa^+)$. For $\nu_0 \preceq \bar{\nu} \leftarrow \tau$, $\{\alpha_\nu\colon \bar{\nu} \leftarrow \nu \leftarrow \tau\}$ is a closed unbounded subset of κ^+. So by $\diamondsuit(\kappa^+)$, for arbitrarily large $\nu \leftarrow \tau$, $X \cap \alpha_\nu = Y_{\alpha_\nu}$. Define B_α as follows:

Case 1. There exist f, τ, ν_0, X as above and ν such that $\nu_0 \preceq \nu \leftarrow \tau$, $\alpha = \alpha_\nu$ and $X \cap \alpha = Y_\alpha$. Set $B_\alpha = \bigcup_{\nu_0 \preceq \bar{\nu} \leftarrow \nu} \pi''_{\bar{\nu}\nu} X_{\bar{\nu}}$. This definition is independent of the choice of f (and thus of τ, ν_0, and X) since $Y_\alpha = \bigcup_{\nu_0 \preceq \bar{\nu} \leftarrow \nu} X_{\bar{\nu}}$ and $X_{\bar{\nu}} = Y_\alpha \cap S_{\alpha_{\bar{\nu}}}$ for $\nu_0 \preceq \bar{\nu} \leftarrow \nu$. Set

$$f_\alpha = \{\langle x, y\rangle\colon (\exists z \in B_\alpha)\, \mathrm{GP}(x, y, z)\}\,.$$

Then $\pi''_{\nu\tau} f_\alpha = f$, $f_\alpha : S_\alpha \to S_\alpha$, type$(\mathrm{dom}(f_\alpha)) = \kappa$, $f_\alpha(x) < x$ for $x \in \mathrm{dom}(f_\alpha)$.

Case 2 (not Case 1). Set $B_\alpha = f_\alpha = 0$. We now use these distinguished functions to define $A_\nu = \langle A_{\nu\gamma}\colon \gamma \in S_{\alpha_\nu} \cap \nu\rangle$ for $\nu \in S$ such that

(0) $A_{\nu\gamma} \subseteq S_{\alpha_\nu} \cap \nu$ & if $\gamma \neq \bar{\gamma}$, then

$$A_{\nu\gamma} \cap A_{\nu\bar{\gamma}} = 0, \qquad \bigcup_{\gamma \in S_{\alpha_\nu} \cap \nu} A_{\alpha\gamma} = S_{\alpha_\nu} \cap \nu\,;$$

(1) if $\bar{\nu} \leftarrow \nu$ & $\bar{\gamma} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$ & $\gamma = \pi_{\bar{\nu}\nu}(\bar{\gamma})$, then $\pi''_{\bar{\nu}\nu} A_{\bar{\nu}\bar{\gamma}} \subseteq A_{\nu\gamma}$;

(2) if ν a limit point in $\leftarrow$ & $\gamma \in S_{\alpha_\nu} \cap \nu$, then

$$A_{\nu\gamma} = \bigcup_{\substack{\bar{\nu} \leftarrow \nu \\ \pi_{\bar{\nu}\nu}(\bar{\gamma}) = \gamma}} \pi''_{\bar{\nu}\nu} A_{\bar{\nu}\bar{\gamma}}\,.$$

Let $\alpha \in A - \{\kappa^+\}$. Let

$$F_\alpha = \{f\colon (\exists \nu \in S_\alpha)(\exists \bar{\nu} \leftarrow \nu)\,(f = \{\langle \pi_{\bar{\nu}\nu}(x),\, \pi_{\bar{\nu}\nu}(f_{\alpha_{\bar{\nu}}}(x))\rangle\colon\ x \in \mathrm{dom}(f_{\alpha_{\bar{\nu}}})\} \;\&\; \mathrm{Card}(f) = \kappa)\}$$

Let G_α be the set of co-initial segments of members of F_α. That is,

$$G_\alpha = \{g\colon (\exists f \in F_\alpha)(\exists \xi < \sup(\mathrm{dom}(f)))(g = f - f \upharpoonright \xi)\}\,.$$

$\mathrm{Card}(G_\alpha) \leqslant \kappa$ and for each $g \in G_\alpha$, $\mathrm{type}(\mathrm{dom}(g)) = \kappa$. Let $\langle g_i^\alpha\colon i \in \theta\rangle$, $\theta \leqslant \kappa$, be an enumeration of G_α. We can choose $\rho_i \in \mathrm{dom}(g_i^\alpha)$ all distinct as follows. Let ρ_0 be any member of $\mathrm{dom}(g_0^\alpha)$. Supposing we have $\{\rho_j \in \mathrm{dom}(g_j^\alpha)\colon j < i\}$ for some $i < \theta$ such that for $j < k < i$, $\rho_j \neq \rho_k$, we can choose $\rho_i \in \mathrm{dom}(g_i^\alpha)$ different from all ρ_j for $j < i$ since $\mathrm{Card}(g_i^\alpha) = \kappa$. We do this for each $\alpha \in A - \{\kappa^+\}$.

By induction on $\nu \in S$, define A_ν satisfying (0)–(2) as follows.

Case 1. ν is minimal in $\leftarrow$. Set $A_{\nu\gamma} = S_{\alpha_\nu} \cap \nu$ if γ is the least member of $S_{\alpha_\nu} \cap \nu$. Otherwise set $A_{\nu\gamma} = 0$.

Case 2. ν is a limit point in $\leftarrow$. Set

$$A_{\nu\gamma} = \bigcup_{\substack{\bar\nu \leftarrow \nu \\ \pi_{\bar\nu\nu}(\bar\gamma) = \gamma}} \pi''_{\bar\nu\nu} A_{\bar\nu\bar\gamma} \quad \text{for each } \gamma \in S_{\alpha_\nu} \cap \nu\,.$$

Case 3. ν has $\bar\nu$ as its immediate predecessor in $\leftarrow$.

Case 3.1. $\bar\nu = \rho_i$ for some $i \in \kappa$, where $\rho_i \in \mathrm{dom}(g_i^{\alpha_{\bar\nu}})$ as defined above For $\gamma \in S_{\alpha_\nu} \cap \nu$ there are three possibilities.

(a) $\gamma = \pi_{\bar\nu\nu}(\bar\gamma)$, where $\bar\gamma = g_i^{\alpha_{\bar\nu}}(\rho_i)$,

(b) $\gamma = \pi_{\bar\nu\nu}(\bar\gamma)$ for some $\bar\gamma \in S_{\alpha_{\bar\nu}} \cap \bar\nu$, $\bar\gamma \neq g_i^{\alpha_{\bar\nu}}(\rho_i)$,

(c) $\gamma \notin \pi''_{\bar\nu\nu} S_{\alpha_{\bar\nu}}$.

In case (a), set $A_{\nu\gamma} = \pi''_{\bar\nu\nu} A_{\bar\nu\bar\gamma} \cup [(S_{\alpha_\nu} \cap \nu) - \pi''_{\bar\nu\nu} S_{\alpha_{\bar\nu}}]$. In case (b), set $A_{\nu\gamma} = \pi''_{\bar\nu\nu} A_{\bar\nu\bar\gamma}$. In case (c) set $A_{\nu\gamma} = 0$.

Case 3.2 (not Case 3.1). For $\gamma \in (S_{\alpha_\nu} \cap \nu) - \pi''_{\bar\nu\nu} S_{\alpha_{\bar\nu}}$ set $A_{\nu\gamma} = 0$.

Otherwise there is $\bar\gamma \in S_{\alpha_{\bar\nu}} \cap \bar\nu$ such that $\pi_{\bar\nu\nu}(\bar\gamma) = \gamma$. Then set $A_{\nu\gamma} = \pi''_{\bar\nu\nu} A_{\bar\nu\bar\gamma}$.

It follows easily from these definitions of A_ν in all three cases that A_ν satisfies conditions (0), (1) and (2) assuming (0), (1) and (2) are satisfied by all $\bar\nu \in S \cap \nu$. This completes the induction. In particular, A_ν for $\nu \in S_{\kappa^+}$ satisfies (0). So it remains to verify the covering part of (*).

Let $f: S_{\kappa^+} \to S_{\kappa^+}$, $\mathrm{type}(\mathrm{dom}(f)) = \kappa$, and $f(x) < x$ for $x \in \mathrm{dom}(f)$. Let $\tau = \sup(\mathrm{dom}(f))$, $B = \{z\colon (\exists x \in \mathrm{dom}(f))\,\mathrm{GP}(x, f(x), z)\}$, $\hat\tau$ the least member of S_{κ^+} greater than B. Then $\tau \leqslant \hat\tau$. (Actually, by invoking further properties of GP and the fact that the $\mathfrak{M}_\nu$ of the Morass are really Jensen's J_ν, we could assert that $\tau = \hat\tau$; but this is not necessary.)

Let $\theta \in S_{\kappa^+}$, $\theta < \mathrm{dom}(f)$. Let $\hat{\nu}_0 \leftarrow \hat{\tau}$ such that $\pi''_{\hat{\nu}_0\hat{\tau}} B_{\alpha_{\hat{\nu}_0}} = B, \theta, \tau \in \pi''_{\hat{\nu}_0\hat{\tau}} S_{\alpha_{\hat{\nu}_0}}$ Let $\nu_0 \in S_{\alpha_{\hat{\nu}_0}}$ such that $\pi_{\hat{\nu}_0\hat{\tau}}(\nu_0) = \tau$. Then $\nu_0 \leqslant \hat{\nu}_0$ and $\pi''_{\nu_0\tau} f_{\alpha_{\nu_0}} = f$. Let $W = (S_{\kappa^+} \cap \theta) - \pi''_{\nu_0\tau} S_{\alpha_{\nu_0}}$. Then $W < \theta$ and W is almost all of $S_{\kappa^+} \cap \theta$ since $\mathrm{Card}(S_{\alpha_{\nu_0}}) = \kappa$. We wish to show that $W \subseteq \bigcup_{x \in \mathrm{dom}(f)} A_{xf(x)}$. Let $a \in W$. Then there is a least $\nu \leftarrow \tau$ such that $a \in \pi''_{\nu\tau} S_{\alpha_\nu}$. ν must have an immediate predecessor in $\leftarrow$, say $\bar{\nu}$. Then $\nu_0 \mathrel{\underline{\leftarrow}} \bar{\nu} \leftarrow \nu \leftarrow \tau$, and there is $a_\nu \in S_{\alpha_\nu} \cap \nu$ such that $\pi_{\nu\tau}(a_\nu) = a$ and for all $\bar{a} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$, $\pi_{\bar{\nu}\nu}(\bar{a}) \neq a_\nu$. Since $\theta \in \pi''_{\nu_0\tau} S_{\alpha_{\nu_0}}$, there exist $\theta_0 \in S_{\alpha_{\nu_0}} \cap \nu_0$, $\bar{\theta} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$, $\theta_\nu \in S_{\alpha_\nu} \cap \nu$ such that $\theta_0 \mathrel{\underline{\leftarrow}} \bar{\theta} \leftarrow \theta_\nu \leftarrow \theta$, $\pi_{\nu_0\tau}(\theta_0) = \theta$, $\pi_{\bar{\nu}\nu}(\bar{\theta}) = \theta_\nu$, etc., $a_\nu \in (S_{\alpha_\nu} \cap \theta_\nu) - \pi''_{\bar{\theta}\theta_\nu} S_{\alpha_{\bar{\nu}}}$, So there is a least δ such that $\bar{\theta} \leftarrow \delta \leftarrow \theta_\nu$, and such that for some $a_\delta \in S_{\alpha_\delta} \cap \delta$, $\pi_{\delta\theta_\nu}(a_\delta) = a_\nu$. From the definition of $G_{\alpha_{\bar{\nu}}}$ it follows that $\pi''_{\nu_0\bar{\nu}} f_{\alpha_{\nu_0}} \in G_{\alpha_{\bar{\nu}}}$. Let $g = \pi''_{\nu_0\bar{\nu}} f_{\alpha_{\nu_0}}$. Then each co-initial segment of g is in $G_{\alpha_{\bar{\nu}}}$. Since $\tau = \sup(\mathrm{dom}(f))$, it follows that

$$\nu_0 = \sup(\mathrm{dom}(f_{\alpha_{\nu_0}}))\,, \qquad \bar{\nu} = \sup(\mathrm{dom}(g))\,,$$

$$\nu = \sup(\mathrm{dom}(\pi''_{\bar{\nu}\nu} g))\,.$$

In particular, then $\nu = \sup \pi''_{\bar{\nu}\nu}(S_{\alpha_{\bar{\nu}}} \cap \bar{\nu})$. By Proposition 2,

$$\alpha_\nu = \sup\{\alpha_{\eta(\nu')} : \nu' \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}\}\,.$$

Also $\alpha_\delta < \alpha_\nu$. So by choosing an appropriate co-initial segment of g, say $\bar{g}$, we can get $\rho_i \in \mathrm{dom}(\bar{g})$ such that $\alpha_{\eta(\rho_i)} > \alpha_\delta$, where $\bar{g} = g_i^{\alpha_{\bar{\nu}}}$ (see Fig. 4). Let $\bar{x} = \rho_i$, $\bar{y} = \bar{g}(\bar{x})$. Then there are $x_0, y_0 \in S_{\alpha_{\nu_0}}$ such that $f_{\alpha_{\nu_0}}(x_0) = y_0$ and $\pi_{\nu_0\bar{\nu}}(x_0) = \bar{x}$, $\pi_{\nu_0\bar{\nu}}(y_0) = \bar{y}$. Let $x' = \eta(\bar{x})$. Let $\delta' = \pi_{\bar{x}x'}(\bar{\theta})$. δ' is on the branch from $\bar{\theta}$ to θ_ν and so is δ. But $\alpha_\delta < \alpha_{x'} < \alpha_{\delta'}$. So $\delta \leftarrow \delta'$. Let $a' = \pi_{\delta\delta'}(a_\delta)$. Then $a' \in (S_{\alpha_{x'}} \cap x') - \pi''_{\bar{x}x'} S_{\alpha_{\bar{x}}}$. Let $y' = \pi_{\bar{x}x'}(\bar{y})$. Then by case 3.1 in the definition of $\langle A_\zeta : \zeta \in S\rangle$ we have, since $x' = \eta(\rho_i)$, that $a' \in A_{x'y'}$. Let $x = \pi_{\bar{\nu}\tau}(\bar{x})$, $y = \pi_{\bar{\nu}\tau}(\bar{y})$. It follows then that $a \in A_{xy}$ and $y = f(x)$.

This completes the proof.

4. Two consequences of the extended Prikry principle

4.1. $(\hat{*})$ *implies* $\omega_{\alpha+2} \not\to [\omega_{\alpha+1} : \omega_\alpha]^2_{\aleph_{\alpha+1}}$

Let $\kappa = \omega_\alpha$. Let $A \otimes B = \{\{a, b\} : a \in A \;\&\; b \in B\}$. We wish to define a partition $I = \langle I_\xi : \xi \in \kappa^+\rangle$ of $[\kappa^{++}]^2$ such that for any $A, B \subseteq \kappa^{++}$ with

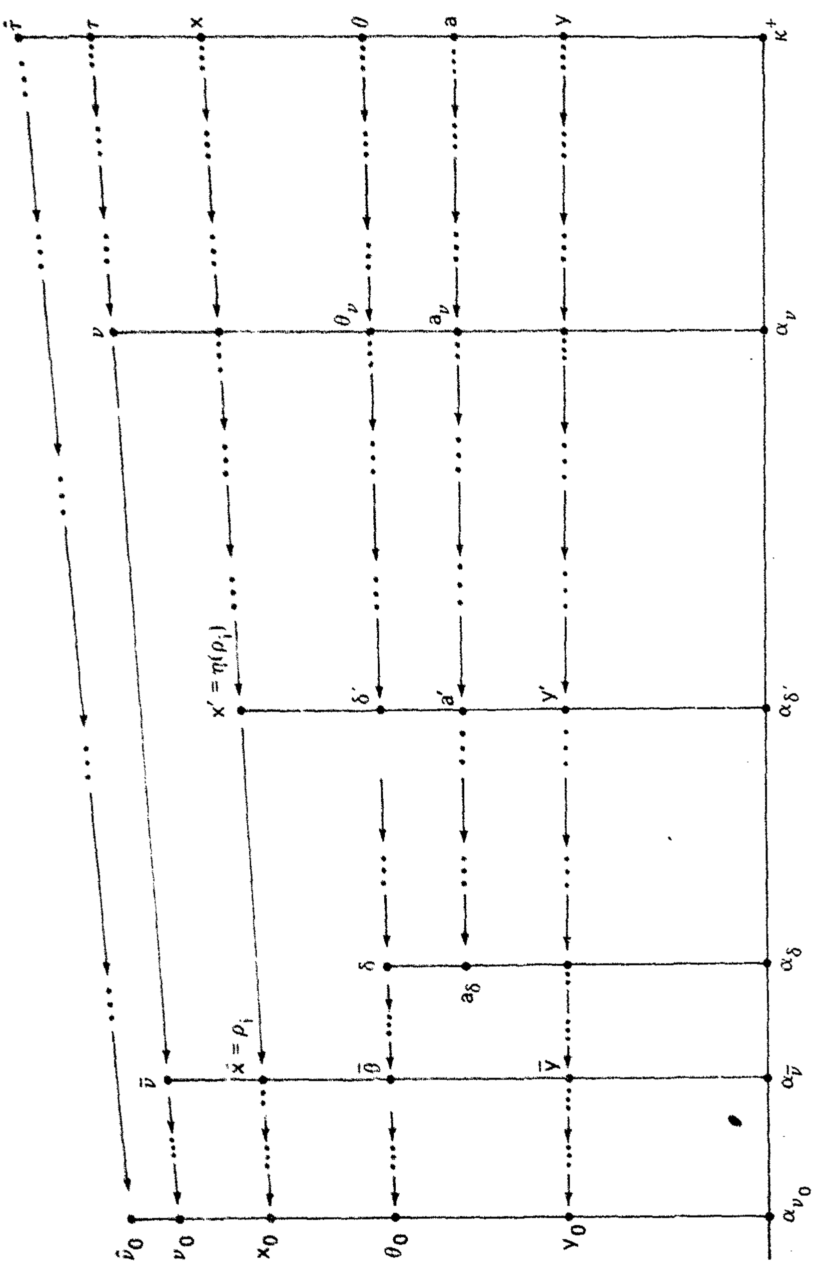
$\hat{\tau}$
τ
x
θ
a
y
κ^+
ν
θ_ν
a_ν
α_ν
$x' = \eta(\rho_1)$
δ'
a'
y'
$\alpha_{\delta'}$
δ
a_δ
α_δ
$\bar{\nu}$
$\bar{x} = \rho_1$
$\bar{\theta}$
$\bar{y}$
$\alpha_{\bar{\nu}}$
$\hat{\nu}_0$
ν_0
x_0
θ_0
y_0
α_{ν_0}

Fig. 4.

$A < B$, type$(A) = \kappa^+$, type$(B) = \kappa$ we have for each $\xi \in \kappa^+$, $I_\xi \cap (A \otimes B) \neq 0$

Let $A_{\beta\xi}$, $\beta \in \kappa^{++}$, $\xi \in \beta$ be given by ($\hat{*}$). Let $I_\xi = \bigcup_{\beta \in \kappa^{++}} (A_{\beta\xi} \otimes \{\beta\})$. Let $A, B \subseteq \kappa^{++}$, $A < B$, type$(A) = \kappa^+$, type$(B) = \kappa$. Let $\theta = \sup(A)$. For $\xi \in \kappa^+$ let $f_\xi : B \to \{\xi\}$. Applying ($\hat{*}$) to f_ξ we have Card$(\theta - \bigcup_{\beta \in B} A_{\beta\xi}) \leqslant \kappa$. But $A \subseteq \theta$ and Card$(A) = \kappa^+$. So there is $\alpha \in A$ and $\beta \in B$ such that $\alpha \in A_{\beta\xi}$, i.e., $\{\alpha, \beta\} \in I_\xi$. Then $I = \langle I_\xi : \xi \in \kappa^+ \rangle$ does it.

Notice that although we only need $\langle I_\xi : \xi < \kappa^+ \rangle$ we have: there is $\langle I_\xi : \xi < \kappa^{++} \rangle$ a partition of $[\kappa^{++}]^2$ such that for any $A, B \subseteq \kappa^{++}$, $A < B$, type$(A) = \kappa^+$ and type$(B) = \kappa$, for each $\xi < \sup(B)$ we have $(A \otimes B) \cap I_\xi \neq 0$.

Also, in the proof of ($\hat{*}$), each co-initial segment of the distinguished function f_α was a member of G_α. From this, it follows that there is $a \in A$ and an increasing sequence $\langle \beta_i : i \in \kappa \rangle$, $\beta_i \in B$ such that $\{a, \beta_i\} \in I_\xi$ for all $i \in \kappa$. That is, for any $\xi < \sup(B)$, there exist arbitrarily large $a \in A$ for which there exist arbitrarily large $\beta \in B$ with $\{a, \beta\} \in I_\xi$.

4.2. ($\hat{*}$) *gives a partial answer to a question of Máté*

Let Y be a set and $f : [Y]^2 \to [Y]^{<\omega}$, where $[Y]^{<\omega}$ is the set of finite subsets of Y. A set $X \subseteq Y$ is said to be free with respect to f if for each $x, y \in X$, $x \neq y$, $f(\{x, y\}) \cap X = 0$. A. Máté asked, in a personal communication to F. Galvin, the following question: Is it true that for any function f with dom$(f) = \{\{\alpha, \beta\} : \alpha < \beta < \omega_2\}$ and $f(\{\alpha, \beta\}) =$ a finite subset of $\{\xi : \alpha < \xi < \beta\}$, there is a set X of cardinality $\aleph_2$ which is a free set with respect to f? A. Hajnal and A. Máté consider various related and similar questions in [7]. In particular see [7, Problems 3 and 4]. Using a result of Erdös and Hajnal, they show that the continuum hypothesis gives a positive answer to Problem 3, which is: "Let $f : [\aleph_2]^2 \to [\aleph_2]^{<\omega}$ be a set mapping such that $\alpha < \bigcap f(\{\alpha, \beta\})$ and $\bigcup f(\{\alpha, \beta\}) < \beta$ hold for any $\alpha < \beta < \aleph_2$. Is there a set of cardinality $\aleph_1$ that is free with respect to f?" Actually a proof similar to that given in [7] would show, assuming the continuum hypothesis, that for any such f there is a free set with respect to f of type $\omega_1 + 1$. Hajnal and Máté also show, among other things, that for $f : [\aleph_2]^2 \to [\aleph_2]^{<\aleph_1}$ such that $\alpha < \bigcap f(\{\alpha, \beta\})$ and $\bigcup f(\{\alpha, \beta\}) \leqslant \beta$ for $\alpha < \beta < \aleph_2$, there is an infinite set free with respect to f.

Using ($\hat{*}$), we can get a function f with dom$(f) = \{\{\alpha, \beta\} : \alpha < \beta < \kappa^{++}\}$

and $f(\{\alpha,\beta\})$ = a finite subset of $\{\xi: \alpha < \xi < \beta\}$ such that no subset of κ^{++} of type $\kappa^+ + \kappa$ is free with respect to f. Specifically we can get $f: [\kappa^{++}]^2 \to [\kappa^{++}]^1$ with $f(\{\alpha,\beta\}) < \beta$ for $\alpha < \beta < \kappa^{++}$ such that for any $A, B \subseteq \kappa^{++}$, with $\mathrm{type}(A) = \kappa^+$, $\mathrm{type}(B) = \kappa$, $A < B$, for any $\xi < \sup(B)$, there are $\alpha \in A$, $\beta \in B$ such that $f(\{\alpha,\beta\}) = \{\xi\}$.

Let I_ξ for $\xi \in \kappa^{++}$ be as above. I.e., $I_\xi = \bigcup_{\alpha\in\kappa^{++}} (A_{\alpha\xi} \otimes \{\alpha\})$. Let $\alpha < \beta < \kappa^{++}$. Set $f(\{\alpha,\beta\}) = \{\xi\}$ if $\{\alpha,\beta\} \in I_\xi$ and $\alpha < \xi < \beta$; $f(\{\alpha,\beta\}) = 0$ otherwise. Let $X \subseteq \kappa^{++}$, $X = A \cup B$, A, B as above. Let $\xi < \sup(B)$. Then $(A \otimes (B - \xi)) \cap I_\xi \neq 0$. In particular, choosing $\xi \in B$ we get that X is not free with respect to f. Similarly we can define $g(\{\alpha,\beta\}) = \{\xi\}$ if $\{\alpha,\beta\} \in I_\xi$ and $\xi < \alpha$; $f(\{\alpha,\beta\}) = 0$ otherwise. Choosing $\xi \in A$ we have $((A - \xi) \otimes B) \cap I_\xi \neq 0$; so X is not free with respect to g (cf. [7, Problem 4])

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